QUESTION 1. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $\int_0^3 \frac{x\,dx}{\sqrt{16+x^2}}$. **3**

(b) Find $\int \frac{dx}{x^2+6x+13}$. **2**

(c) Find $\int xe^{-x}\,dx$. **2**

(d) Find $\int \cos^3\theta\,d\theta$. **3**

(e) (i) Find constants A, B and C such that **3**

$$\frac{x^2-4x-1}{(1+2x)(1+x^2)} \equiv \frac{A}{1+2x} + \frac{Bx+C}{1+x^2}.$$

(ii) Hence find $\int \frac{x^2-4x-1}{(1+2x)(1+x^2)}\,dx$. **2**

QUESTION 2. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Given that $z = 1 + i$ and $w = -3$, find, in the form $x + iy$:

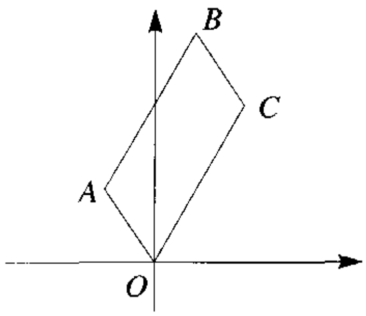
(i) wz^2, **1**

(ii) $\dfrac{z}{z + w}$. **2**

(b) Using de Moivre's theorem, simplify $(-1 - i\sqrt{3})^{-10}$, expressing the answer in the form $x + iy$. **3**

(c) Find the values of real numbers a and b such that $(a + ib)^2 = 5 - 12i$. **2**

(d) **3**



In the diagram above, $OABC$ is a parallelogram with $OA = \frac{1}{2}OC$.

The point A represents the complex number $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

If $\angle AOC = 60^\circ$, what complex number does C represent?

(e) z_1 and z_2 are complex numbers.

(i) Show that $|z_1||z_2| = |z_1 z_2|$. **1**

(ii) By taking $z_1 = 2 + 3i$ and $z_2 = 4 + 5i$, express 533 (the product of 13 and 41) as a sum of squares of two positive integers. **1**

(iii) By taking other values for z_1 and z_2, express 533 as a sum of squares of two other positive integers. **2**

QUESTION 3. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) *On separate number planes, draw graphs of the following functions, showing essential features.*

(i) $y = \frac{x+1}{x-1}$ **2**

(ii) $y = \sqrt{\frac{x+1}{x-1}}$ **2**

(iii) $y = \ln\left(\frac{x+1}{x-1}\right)$ **2**

(b) z is a variable complex number which is represented by the point P. Find the locus of P if $|z - 2i| = \text{Im}(z)$ **2**

(c) The fixed complex number α is such that $0 < \arg\alpha < \frac{\pi}{2}$. In an Argand diagram α is represented by the point A while $i\alpha$ is represented by the point B. z is a variable complex number which is represented by the point P.

(i) Draw a diagram showing A, B and the locus of P if $|z-\alpha| = |z - i\alpha|$. **1**

(ii) Draw a diagram showing A, B and the locus of P if $\arg(z-\alpha) = \arg(i\alpha)$. **1**

(iii) Find in terms of α the complex number represented by the point of intersection of the two loci in (i) and (ii). **1**

(d) Consider the function $y = \sin^{-1}(e^x)$.

(i) Find the domain and range of the function. **2**

(ii) Sketch the graph of the function showing clearly the coordinates of any endpoints and the equations of any asymptotes. **2**

QUESTION 4. (15 marks) Use a SEPARATE writing booklet. **Marks**

Consider the ellipse $\frac{x^2}{16} + \frac{y^2}{12} = 1$.

(a) (i) Find the eccentricity of the ellipse. **1**

(ii) Find the coordinates of the foci and the equations of the directrices of the ellipse. **2**

(iii) Sketch the graph of the ellipse showing clearly all of the above features and the intercepts on the coordinate axes. **2**

(b) (i) Use differentiation to derive the equations of the tangent and the normal to the ellipse at the point $P(2,3)$. **3**

(ii) The tangent and normal to the ellipse at P cut the y axis at A and B respectively. Find the coordinates of A and B. **1**

(c) (i) Show that AB subtends a right angle at the focus S of the ellipse. **2**

(ii) Show that the points A, P, S and B are concyclic. **1**

(iii) Find the centre and radius of the circle which passes through the points A, P, S and B. **3**

QUESTION 5. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Let $P(x)$ be a degree 4 polynomial with a zero of multiplicity 3. Show that $P'(x)$ has a zero of multiplicity 2. **2**

(ii) Hence or otherwise find all zeros of $P(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, given that it has a zero of multiplicity 3. **2**

(iii) Sketch $y = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, clearly showing the intercepts on the coordinate axes. You do not need to give the coordinates of turning points or inflections. **1**

(b) (i) Show that the general solution of the equation $\cos 5\theta = -1$ is given by $\theta = (2n+1)\frac{\pi}{5}$, $n = 0, \pm 1, \pm 2, \ldots$. **2**

Hence solve the equation $\cos 5\theta = -1$ for $0 \le \theta \le 2\pi$.

(ii) Use De Moivre's Theorem to show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. **3**

(iii) Find the exact trigonometric roots of the equation $16x^5 - 20x^3 + 5x + 1 = 0$. **2**

(iv) Hence find the exact values of $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}$ and $\cos\frac{\pi}{5} . \cos\frac{3\pi}{5}$ and factorise $16x^5 - 20x^3 + 5x + 1$ into irreducible factors over the rational numbers. **3**

QUESTION 6. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) A lifebelt mould is made by rotating the circle $x^2 + y^2 = 64$ through one complete revolution about the line $x = 28$, where all the measurements are in centimetres.

(i) Use the method of slicing to show that the volume $V\text{cm}^3$ of the lifebelt is given by **5**

$$V = 112\pi \int_{-8}^{8} \sqrt{64 - y^2}\ dy.$$

(ii) Find the exact volume of the lifebelt. **2**

(b) It is given that if a, b, c are any three positive real numbers, then $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$.

If $a > 0$, $b > 0$ and $c > 0$ are real numbers such that $a + b + c = 1$, use the given result to show that

(i) $\frac{1}{abc} \geq 27$ **1**

(ii) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9$ **2**

(iii) $(1-a)(1-b)(1-c) \geq 8abc$ **2**

(c) In a series of five games played by two equally matched teams, team A and team B, the team that wins three games first is the champion.

(i) If team B wins the first two games, what is the probability that team A is the champion? **1**

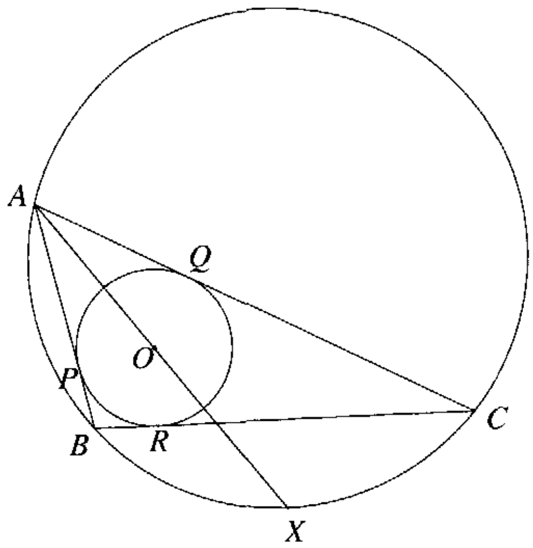
(ii) If team A has won the first game, what is the probability that team A is the champion? **2**

QUESTION 7. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) In the diagram below, ABC is a triangle.

The incircle tangent to all three sides has centre O, and touches the sides AB, AC and BC at P, Q and R respectively.

The circumcircle through A, B and C meets the line AO produced at X.



(i) Show that $\angle CBX = \angle CAX$. **1**

(ii) Use congruence to prove that $\angle OBA = \angle OBC$. **2**

(iii) Prove that ΔXBO is an isosceles triangle. **3**

(iv) Prove that $BX = XC$. **1**

(b) (i) α. Differentiate $y = \log_e(1+x)$, and hence draw $y = x$ and $y = \log_e(1+x)$ on one set of axis. **1**

β. Using this graph, explain why **1**

$$\log_e(1+x) < x, \text{ for all } x > 0.$$

(ii) α. Differentiate $y = \frac{x}{1+x}$, and hence draw $y = \frac{x}{1+x}$ and $y = \log_e(1+x)$ on one set of axis. **1**

β. Using this graph, explain why **1**

$$\frac{x}{1+x} < \log_e(1+x), \text{ for all } x > 0.$$

(iii) Use the inequalities of parts (i) and (ii) to show that **4**

$$\frac{\pi}{8} - \frac{1}{4}\log_e 2 < \int_0^1 \frac{\log_e(1+x)}{1+x^2}\,dx < \frac{1}{2}\log_e 2.$$

QUESTION 8. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) At a dinner party there are twelve people, consisting of the six State Premiers and their partners. Each couple was representing one of the six States: New South Wales, Victoria, Western Australia, South Australia, Tasmania and Queensland.

(i) The dinner took place at a circular table. Find how many seating arrangements are possible if:

α. there are no restrictions, **1**

β. the males and females are in alternate positions. **1**

(ii) A committee of six is to be formed from the Premiers and their partners, where not more than one State can have two representatives. How many such committees are possible? **2**

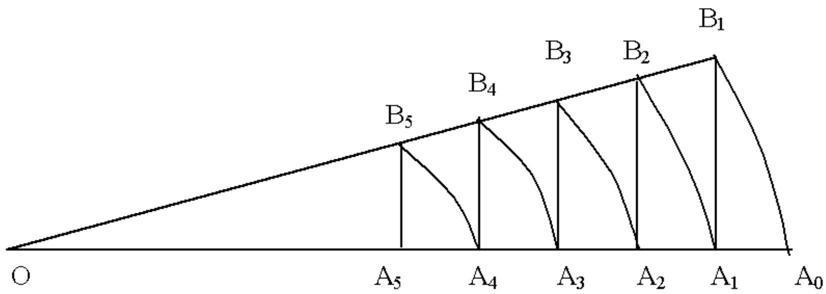
(b) (i) Show that $\dfrac{t^n}{1+t^2} = t^{n-2} - \dfrac{t^{n-2}}{1+t^2}$. **1**

(ii) Let $I_n = \displaystyle\int \frac{t^n}{1+t^2}\,dt$. **1**

Show that $I_n = \dfrac{t^{n-1}}{n-1} - I_{n-2},\ n \geq 2$.

(iii) Show that $\displaystyle\int_0^1 \frac{t^6}{1+t^2}\,dt = \frac{13}{15} - \frac{\pi}{4}$. **3**

(c)



An ant walks along the circular arc from A_0 to B_1, then down the straight line to A_1, along the circular arc to B_2, then down to A_2, and so on, until it reaches O.

The length of OA_0 is 1, while angle A_0OB_1 is x radians, $0 < x \leq \dfrac{\pi}{2}$.

(i) Show that the total distance the ant walks by the time it reaches O is given by $y = \dfrac{x + \sin x}{1 - \cos x}$ **2**

(ii) Find the derivative of y with respect to x and explain why the derivative of y is always negative for all $0 < x \leq \dfrac{\pi}{2}$ **2**

(iii) Hence find the shortest possible distance the ant needs to walk from A_0 to O. **2**

End of Paper

QUESTION 1

(a) $\displaystyle\int_0^3 \frac{x\,dx}{\sqrt{16+x^2}} = \int_{16}^{25} \frac{\frac{1}{2}\,du}{\sqrt{u}}$

Let $u = 16 + x^2$

$du = 2x\,dx$ ✓

$\displaystyle = \frac{1}{2}\int_{16}^{25} u^{-\frac{1}{2}}\,du$

$\displaystyle = \frac{1}{2}\left[2u^{\frac{1}{2}}\right]_{16}^{25}$ ✓

$\displaystyle = \left[\sqrt{u}\right]_{16}^{25}$

$= \sqrt{25} - \sqrt{16}$

$= 5 - 4$

$= 1$ ✓

(b) $\displaystyle\int \frac{dx}{x^2+6x+13} = \int \frac{dx}{(x+3)^2+4}$ ✓

$\displaystyle = \frac{1}{2}\tan^{-1}\frac{x+3}{2} + c$ ✓

(c) $\displaystyle\int xe^{-x}dx = \int x\frac{d}{dx}(-e^{-x})dx$

$\displaystyle = -xe^{-x} - \int 1(-e^{-x})dx$ ✓

$\displaystyle = -xe^{-x} + \int e^{-x}dx$

$= -xe^{-x} - e^{-x} + c$ ✓

(d) $\displaystyle\int \cos^3\theta\,d\theta = \int \cos^2\theta\cos\theta\,d\theta$ ✓

Let $u = \sin\theta$

$du = \cos\theta\,d\theta$

$\displaystyle = \int (1-\sin^2\theta)\cos\theta\,d\theta$

$\displaystyle = \int (1-u^2)du$ ✓

$\displaystyle = u - \frac{1}{3}u^3 + c$

$\displaystyle = \sin\theta - \frac{1}{3}\sin^3\theta + c$ ✓

(e) (i) Let $\frac{x^2-4x-1}{(1+2x)(1+x^2)} \equiv \frac{A}{1+2x} + \frac{Bx+C}{1+x^2}$.

Then $x^2-4x-1 \equiv A(1+x^2)+(1+2x)(Bx+C)$ ✓

$x^2-4x-1 \equiv A+Ax^2+Bx+C+2Bx^2+2Cx$

Equating coefficients of like terms,

$1 = A+2B$ [Eq. 1]

$-4 = B+2C$ [Eq. 2]

$-1 = A+C$ [Eq. 3]

Multiply Eq. 2 by -2:

$8 = -2B-4C$ [Eq. 2a]

Eq. 1 + Eq. 2a:

$9 = A-4C$ [Eq. 4]

Eq. 3 − Eq. 4:

$-10 = 5C$

$C = -2$

Substitute C into Eq. 3:

$-1 = A-2$

$A = 1$

Substitute A into Eq. 1:

$1 = 1+2B$

$B = 0$ ✓✓

(ii) $\int \frac{x^2-4x-1}{(1+2x)(1+x^2)}\,dx = \int\left(\frac{1}{1+2x} + \frac{-2}{1+x^2}\right)dx$

$= \frac{1}{2}\ln|1+2x| - 2\tan^{-1}x + c$ ✓✓

QUESTION 2

(a) (i) $wz^2 = -3(1+i)^2$

$= -3(1-1+2i)$

$= -6i$ ✓

(ii) $\frac{z}{z+w} = \frac{1+i}{-2+i} \times \frac{-2-i}{-2-i}$

$= \frac{-(1+i)(2+i)}{5}$ ✓

$= \frac{-(1+3i)}{5}$

$= -\frac{1}{5} - \frac{3i}{5}$ ✓

QUESTION 2

(b) $-1-i\sqrt{3}=2\text{cis}\left(-\frac{2\pi}{3}\right)$ ✓

Hence $\left(2\text{cis}\left(-\frac{2\pi}{3}\right)\right)^{-10}=2^{-10}\text{cis}\left(\frac{20\pi}{3}\right)$ ✓

$$=2^{-10}\text{cis}\left(\frac{2\pi}{3}\right)$$

$$=\frac{1}{1024}\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)$$

$$=-\frac{1}{2048}+\frac{i\sqrt{3}}{2048}$$ ✓

(c)

$(a+ib)^2=5-12i \Rightarrow (a^2-b^2)+2ab\,i=5-12i$

$\therefore a^2-b^2=5$ and $ab=-6$

$a^4-a^2b^2=5a^2 \Rightarrow a^4-5a^2-36=0$

$(a^2+4)(a^2-9)=0 \quad \therefore a^2>0 \Rightarrow a^2=9$

$\therefore \begin{cases} a=3 \\ b=-2 \end{cases}$ or $\begin{cases} a=-3 \\ b=2 \end{cases}$

Criteria
• one mark for equating real and imaginary parts • one mark for values of a and b

(d) To rotate $\overrightarrow{OA}$ by $-60°$, we need to multiply by $\text{cis}\left(-\frac{\pi}{3}\right)$ ✓.

Thus $\overrightarrow{OC}=2\times\overrightarrow{OA}\times\text{cis}\left(-\frac{\pi}{3}\right)$

$$=2\times\text{cis}\left(\frac{2\pi}{3}\right)\times\text{cis}\left(-\frac{\pi}{3}\right)$$

$$=2\text{cis}\left(\frac{\pi}{3}\right)$$ ✓

$$=1+i\sqrt{3}$$ ✓

(e)

Criteria	Marks
(i) • one mark for answer	1
(ii) • one mark for answer	1
(iii) • one mark for choice of $z_1,\ z_2$ • one mark for answer	2

(i)

$z_1=a+ib,\ \ z_2=c+id \Rightarrow z_1z_2=(ac-bd)+i(ad+bc)$

$|z_1z_2|^2=(ac-bd)^2+(ad+bc)^2=a^2c^2-2acbd+b^2d^2+a^2d^2+2adbc+b^2c^2$

$\therefore |z_1z_2|^2=(a^2+b^2)(c^2+d^2)=|z_1|^2.|z_2|^2 \qquad \therefore |z_1|.|z_2|=|z_1z_2|$

(ii)

$z_1=2+3i \quad \Rightarrow |z_1|^2=4+9=13$

$z_2=4+5i \quad \Rightarrow |z_2|^2=16+25=41$

$z_1.z_2=-7+22i \Rightarrow |z_1.z_2|^2=7^2+22^2$

$\therefore 533 = 13 \times 41 = 7^2+22^2$

(iii)

For example :

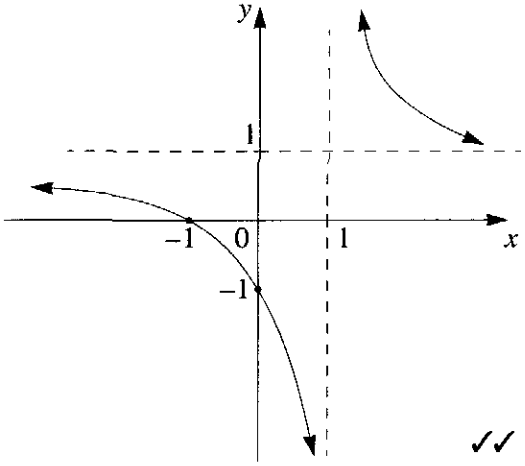
$z_1=3+2i,\ \ z_2=5-4i,\ \ z_1.z_2=23-2i$

$|z_1|^2=13,\ \ |z_2|^2=41,\ \ |z_1.z_2|^2=23^2+2^2$

$\therefore 533=13\times 41=23^2+2^2$

QUESTION 3

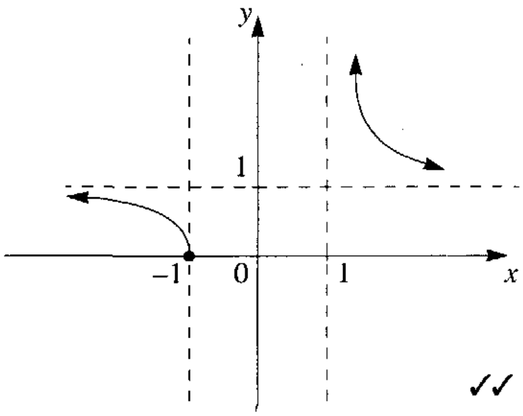
(a) (i)



✓✓

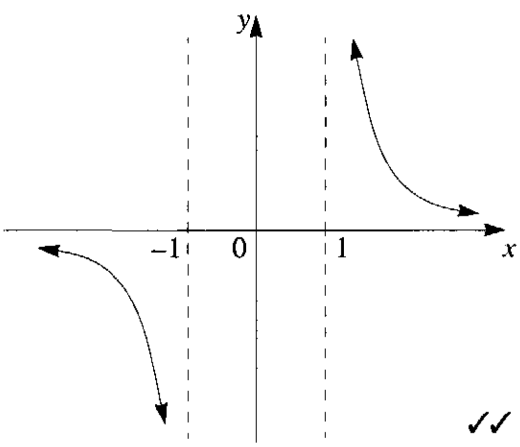
- vertical asymptote at $x = 1$
- horizontal asymptote at $y = 1$
- x intercept at $x = -1$
- y intercept at $y = -1$

(ii)



✓✓

(iii)



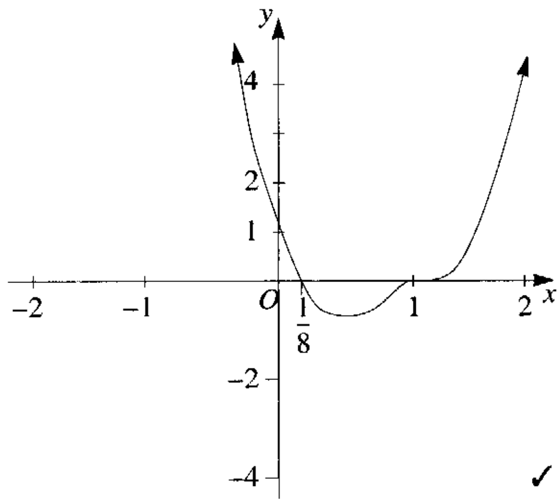
✓✓

(b) The locus is the parabola with focus at (0, 2) ✓ and the x axis as directrix. ✓

or give the equation of the parabola as $x^2 = 4(y-1)$

QUESTION 5

(a) (iii)



5(b)

Marking Guidelines

Criteria	Marks
(i) • one mark for general solution • one mark for particular solution	2
(ii) • one mark for expression for $\text{Re}(\cos\theta + i\sin\theta)^5$ in terms of $\cos\theta$, $\sin\theta$ • one mark for expression for $\text{Re}(\cos\theta + i\sin\theta)^5$ in terms of $\cos\theta$ • one mark for final answer	3
(iii) • one mark for noting that $x = \cos\theta$ where $\cos 5\theta = -1$ • one mark for solution	2
(iv) • one mark for value of $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}$ • one mark for value of $\cos\frac{\pi}{5} . \cos\frac{3\pi}{5}$ • one mark for factorisation	3

Answer

(i)

$\cos 5\theta = -1 \Rightarrow 5\theta = (2n+1)\pi$

$\theta = (2n+1)\frac{\pi}{5}$, $n = 0, \pm 1, \pm 2, \ldots$

$0 \le \theta \le 2\pi \Rightarrow \theta = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$

(ii) Using the binomial expansion,

$\text{Re}\left\{(\cos\theta + i\sin\theta)^5\right\}$

$= \cos^5\theta + 10\cos^3\theta\,(i\sin\theta)^2 + 5\cos\theta\,(i\sin\theta)^4$

$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$= \cos^5\theta - 10\cos^3\theta\,(1-\cos^2\theta) + 5\cos\theta\,(1-\cos^2\theta)^2$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

Using De Moivre's Theorem,

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

Hence $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(iii) $16x^5 - 20x^3 + 5x + 1 = 0$

has solutions $x = \cos\theta$ where $\cos 5\theta = -1$.

$x = \cos\frac{\pi}{5}, \cos\frac{3\pi}{5}, \cos\pi, \cos\frac{7\pi}{5}, \cos\frac{9\pi}{5}$

$x = \cos\frac{\pi}{5}, \cos\frac{\pi}{5}, \cos\frac{3\pi}{5}, \cos\frac{3\pi}{5}, -1$

(iv)

$\sum\alpha = 0 \Rightarrow 2\left(\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}\right) - 1 = 0$

$\therefore \cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$

Product of roots is $-\frac{1}{16}$

$\therefore -\left(\cos\frac{\pi}{5} . \cos\frac{3\pi}{5}\right)^2 = -\frac{1}{16}$

$\therefore \cos\frac{\pi}{5} . \cos\frac{3\pi}{5} = -\frac{1}{4}$

$\left(\text{since } \cos\frac{\pi}{5} > 0, \ \cos\frac{3\pi}{5} < 0\right)$

Then $\cos\frac{\pi}{5}$, $\cos\frac{3\pi}{5}$ are roots of the equation $4x^2 - 2x - 1 = 0$. Hence

$16x^5 - 20x^3 + 5x + 1 = (x+1)(4x^2 - 2x - 1)^2$

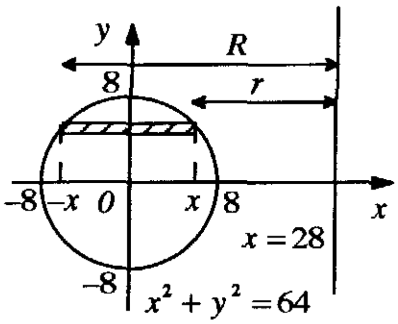
Question 6

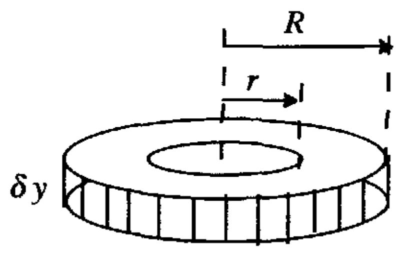
Marking Guidelines

(a)

Criteria	Marks
(i) • one mark for identifying slice as annular prism, thickness δy • one mark for inner radius r in terms of y • one mark for outer radius R in terms of y • one mark for simplified value of δV in terms of y • one mark for expression for V	5
(ii) • one mark for using area of semi circle, or appropriate integration process • one mark for final answer	2

(i)





$$R = 28 + \sqrt{64 - y^2}$$

$$r = 28 - \sqrt{64 - y^2}$$

Volume of slice is

$$\delta V = \pi\left(R^2 - r^2\right)\delta y$$

$$= \pi(R+r)(R-r)\delta y$$

$$= \pi \, . \, 56 \, . \, 2\sqrt{64 - y^2} \, . \, \delta y$$

$$V = \lim_{\delta y \to 0} \sum_{y=-8}^{8} 112\pi \sqrt{64-y^2} \, . \, \delta y$$

$$= 112\pi \int_{-8}^{8} \sqrt{64-y^2} \; dy$$

(ii) $\int_{-8}^{8} \sqrt{64-y^2} \; dy = \frac{1}{2}\pi . 8^2 = 32\pi$ (*Area of semicircle radius* 8) $\Rightarrow$ $V = 3584\pi^2$

Exact volume of lifebelt is $3584\pi^2$ cm^3

(b) (i) $\text{RHS} = t^{n-2} - \dfrac{t^{n-2}}{1+t^2}$

$$= \frac{(1+t^2)t^{n-2} - t^{n-2}}{1+t^2}$$

$$= \frac{t^{n-2} + t^n - t^{n-2}}{1+t^2}$$

$$= \frac{t^n}{1+t^2}$$

$= \text{LHS}$ ✓

(ii) $I_n = \displaystyle\int \frac{t^n}{1+t^2} dt$

$$= \int \left(t^{n-2} - \frac{t^{n-2}}{1+t^2}\right) dt$$

$$= \frac{t^{n-1}}{n-1} - \int \frac{t^{n-2}}{1+t^2} dt$$

$= \dfrac{t^{n-1}}{n-1} - I_{n-2}$ ✓

(iii) Let $J_n = \displaystyle\int_0^1 \frac{t^n}{1+t^2} dt$

Then $J_n = \left[\dfrac{t^{n-1}}{n-1}\right]_0^1 - J_{n-2}$

$= \dfrac{1}{n-1} - J_{n-2}$ ✓

Hence $J_6 = \dfrac{1}{5} - J_4$

$$= \frac{1}{5} - \frac{1}{3} + J_2$$

$= \dfrac{1}{5} - \dfrac{1}{3} + 1 - J_0$ ✓

But $J_0 = \displaystyle\int_0^1 \frac{1}{1+t^2} dt$

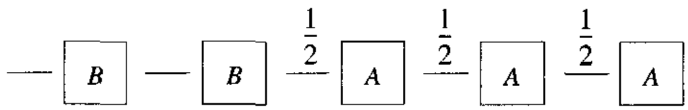
$$= \left[\tan^{-1} t\right]_0^1 = \frac{\pi}{4}$$

Hence $J_6 = \dfrac{1}{5} - \dfrac{1}{3} + 1 - \dfrac{\pi}{4}$

$= \dfrac{13}{15} - \dfrac{\pi}{4}$ ✓

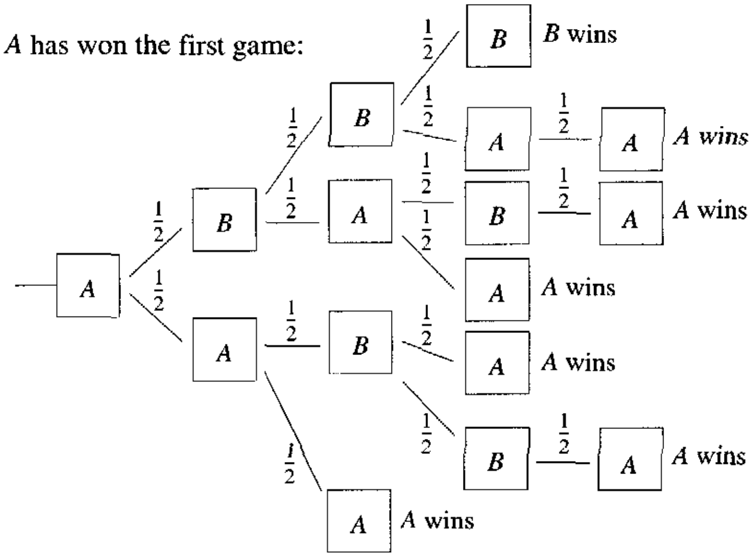
6 (c) The probability that A wins any game is $\frac{1}{2}$.

(i) If team B wins the first two games:



$P(A \text{ wins}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ ✓

(ii) If team A has won the first game:

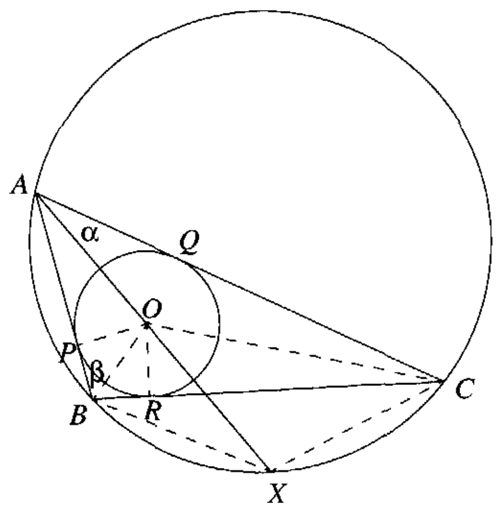


$$P(A \text{ wins}) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2} \quad ✓$$

$$= \frac{1}{16}+\frac{1}{16}+\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\frac{1}{4} = \frac{11}{16} \quad ✓$$

QUESTION 7

(a)



(i) $\therefore \angle CBX = \angle CAX$ (angles on the same arc CX) ✓

(ii) In the triangles POB and ROB:

1. $OB = OB$ (common)
2. $OP = OR$ (radii)
3. $\angle OPB = \angle ORB = 90°$ (radius and tangent)

so $\Delta POB \equiv \Delta ROB$ (RHS) ✓

Hence $\angle OBA = \angle OBC$ (corresponding angles of congruent triangles) ✓

(iii) Let $\angle OBA = \angle OBC = \beta$ and $\angle CAX = \angle CBX = \alpha$.

Then by a similar proof to (ii), $\angle BAX = \alpha$. ✓

Hence $\angle BOX = \alpha + \beta$ *(exterior angle of* ΔABO*).* ✓

But $\angle OBX = \alpha + \beta$ (adjacent angles),

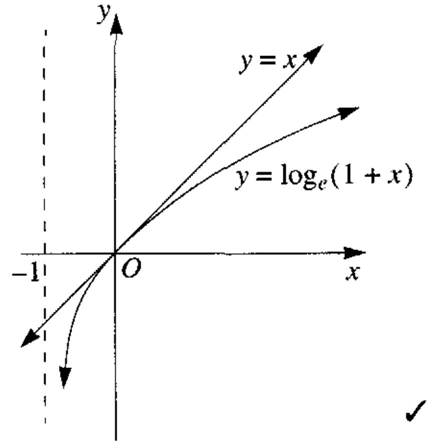
so $BX = OX$ (opposite angles in ΔOBX are equal). ✓

(iv) Similarly, $CX = OX$.

Hence $BX = CX$. ✓

7 (b) (i) α. $y = \log_e(1+x)$

$\dfrac{dy}{dx} = \dfrac{1}{1+x}$



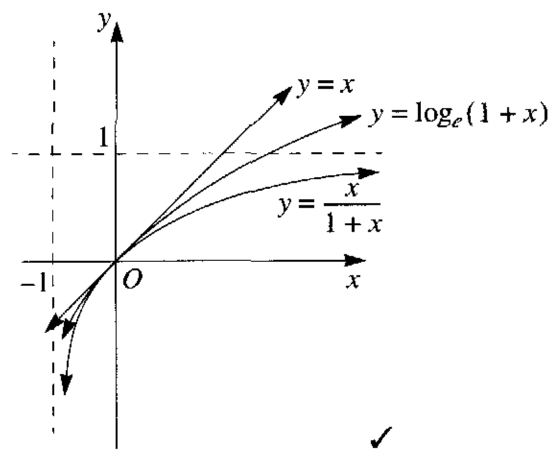
✓

β. When $x = 0$, $\dfrac{dy}{dx} = 1$, so $y = x$ is a tangent at $(0, 0)$.

Since $y = \log_e(1+x)$ is concave down, it follows that its graph is below the line $y = x$ for $x > 0$. ✓

(ii) α. $y = \dfrac{x}{1+x}$

Using the quotient rule, $\dfrac{dy}{dx} = \dfrac{1}{(1+x)^2}$.



✓

β. When $x = 0$, $\dfrac{dy}{dx} = 1$, so $y = x$ is a tangent to both curves at $(0, 0)$.

But for $x > 0$, the gradient function of $y = \dfrac{x}{1+x}$ is less than the gradient function of $y = \log_e(1+x)$, because $\dfrac{1}{(1+x)^2} < \dfrac{1}{1+x}$ for $x > 0$.

Hence the graph of $y = \dfrac{x}{1+x}$ is always below the graph of $y = \log_e(1+x)$ for $x > 0$. ✓

(iii) From (i) and (ii), $\dfrac{x}{1+x} < \log_e(1+x) < x$ for all $x > 0$.

Hence $\dfrac{x}{(1+x)(1+x^2)} < \dfrac{\log_e(1+x)}{1+x^2} < \dfrac{x}{1+x^2}$ for all $x > 0$

and so $\displaystyle\int_0^1 \frac{x}{(1+x)(1+x^2)}\,dx < \int_0^1 \frac{\log_e(1+x)}{1+x^2}\,dx < \int_0^1 \frac{x}{1+x^2}\,dx$ for all $x > 0$. ✓

Now $\displaystyle\int_0^1 \frac{x}{1+x^2}\,dx = \left[\frac{1}{2}\log_e(x^2+1)\right]_0^1$

$= \dfrac{1}{2}\log_e 2$ ✓

7 (b) (iii)

(cont'd) Also, $\int_0^1 \frac{x}{(1+x)(1+x^2)}\,dx = \int_0^1 \left(-\frac{1}{2(x+1)} + \frac{1+x}{2(x^2+1)}\right)dx$ (partial fractions)

$$= \left[-\frac{1}{2}\log_e(1+x)\right]_0^1 + \left[\frac{1}{4}\log_e(x^2+1)\right]_0^1 + \left[\frac{1}{2}\tan^{-1}x\right]_0^1 \quad ✓$$

$$= -\frac{1}{2}\log_e 2 + \frac{1}{4}\log_e 2 + \frac{1}{2}\tan^{-1}1$$

$$= \frac{\pi}{8} - \frac{1}{4}\log_e 2$$

Hence $\frac{\pi}{8} - \frac{1}{4}\log_e 2 < \int_0^1 \frac{\log_e(1+x)}{1+x^2}\,dx < \frac{1}{2}\log_e 2$ for all $x > 0$. ✓

QUESTION 8

(a) (i) α. Number of arrangements when there are no restrictions $= 11!$ ✓

$= 39916800$

β. The males and females are in alternate positions.

Sit a person down. There are $5!$ ways of seating the remaining members of the same sex.

Then there are $6!$ ways of seating the opposite sex.

So the total number of ways $= 5! \times 6!$ ways. ✓

(ii) Two cases:

(1) If one state has two representatives, number of ways $= \binom{6}{4} \times 2^5 = 480$ ✓

(2) If no state has two representatives, number of ways $= 2^6 = 64$ ✓

Hence total number of ways = 480 + 64 = 544

(b)

Criteria	Marks
(i) • one mark for answer	1
(ii) • one mark for replacing a, b, c by $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ respectively • one mark for final answer	2
(iii) • one mark for use of $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9$ • one mark for final answer	2

(i)

$$\sqrt[3]{abc} \leq \frac{a+b+c}{3} = \frac{1}{3}$$

$$abc \leq \frac{1}{27}$$

$$\frac{1}{abc} \geq 27$$

(ii)

$$\frac{1}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq \sqrt[3]{\left(\frac{1}{a}\right)\left(\frac{1}{b}\right)\left(\frac{1}{c}\right)}$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 3\sqrt[3]{\frac{1}{abc}}$$

$$\geq 3\sqrt[3]{27}$$

$$\therefore \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9$$

(iii)

$$(1-a)(1-b)(1-c)$$

$$= 1 - (a+b+c) + (bc+ca+ab) - abc$$

$$= (bc+ca+ab) - abc$$

$$= abc\left\{\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 1\right\}$$

$$\geq abc\,(9-1)$$

$$\therefore (1-a)(1-b)(1-c) \geq 8\,abc$$